

Non-degenerate forms and dual spaces

1 Dual space representation theorem

Let V be a vector space over a field k .

Definition 1.1. A symmetric bilinear form on V is a bilinear map $(\cdot, \cdot) : V \times V \rightarrow k$ satisfying

$$(u, v) = (v, u), \text{ for all } u, v \in V.$$

The form is called **non-degenerate** if

$$(u, v) = 0, \text{ for all } v \in V \text{ implies } u = 0.$$

From now on equip V with a symmetric form $(\cdot, \cdot)$.

Definition 1.2. For each $u \in V$ we define the map $m_u : V \rightarrow k$, by

$$m_u(v) = (u, v).$$

Proposition 1.3. The map sending $u \mapsto m_u$ is linear map from $V \rightarrow V^*$.

Proof. By definition $m_u \in V^*$. For linearity it suffices to check that

$$\begin{aligned} m_{au} &= am_u \\ m_{u+v} &= m_u + m_v \end{aligned}$$

both of which follow from linearity in the first coordinate for $(\cdot, \cdot)$. □

Theorem 1.4 (Representation theorem for V^*). Let V be finite dimensional, and suppose the bilinear form is non-degenerate. Then for every $f \in V^*$ there is a $u \in V$ such that

$$f(v) = (u, v), \text{ for all } v \in V.$$

That is all functionals are induced from the bilinear form. More formally the map $\psi(u) = m_u$ is an isomorphism.

Proof. Recall that the map $\psi : V \rightarrow V^*$ defined by $\psi(u) = m_u$ is linear. Then the first isomorphism theorem yields

$$\frac{V}{\ker \psi} \cong \text{im } \psi \subseteq V^*.$$

Now consider that $\psi(u) = m_u$ is the zero map iff

$$m_u(v) = (u, v) = 0, \text{ for all } v \in V.$$

Then since the form is non-degenerate this implies $u = 0$. Then ψ is injective and identifies V as a subspace of V^* . But then since V is finite dimensional $V \cong V^*$ so ψ must be surjective, therefore an isomorphism. □

1.1 Representation restricts to a subspace

Proposition 1.5. Let V be finite dimensional and let $W \subseteq V$ be a subspace. Then we may identify W^* with a subspace of V^* .

Proof. Let $\{w_i\}_{i \in I}$ be a basis for W . Since W is a vector space we may define a map by defining its image on a basis. Then consider the maps $f_i : W \rightarrow k$ defined by

$$f_i(w_j) = \begin{cases} 1, & i = j \\ 0, & \text{else} \end{cases}.$$

It's straight forward to check these form a basis for the dual space W^* (from linear combinations of these maps we can construct any possible image for the basis $\{w_i\}_{i \in I}$ that we want, and any functional is determined by its image on this basis.) Now extend $\{w_i\}$ to a basis for V by $\{w_i\}_{i \in I} \cup \{v_j\}_{j \in J}$. Then a basis for V^* is

$$\{f_i\}_{i \in I} \cup \{g_j\}_{j \in J}$$

thus we have $W^* = \text{span}\{f_i\} \subseteq \text{span}(\{f_i\} \cup \{g_j\}) = V^*$. $\square$

Corollary 1.6. When V is finite dimensional, the form is non-degenerate, and $W \subseteq V$ is a subspace the map $\phi(u) = m_u \upharpoonright W$ is surjective onto W^* .

Proof. If $\{f_i\}_{i \in I}$ is a basis for W^* then each f_i extends to a functional on V^* (since the basis for W extends to a basis for V .) Then there is $u_i \in V$ such that $f_i = m_{u_i}$ for each $i \in I$. Then restricting $f_i = m_{u_i} \upharpoonright W$ so we obtain a basis for the dual space W^* and the map is surjective. $\square$

2 Dimension of orthogonal subspace.

Definition 2.1. Let $W \subseteq V$ be a subspace. Then

$$W^\perp = \{u \in V : (u, w) = 0, \text{ for all } w \in W\}.$$

Proposition 2.2. If V is finite dimensional and the bilinear form is non-degenerate then

$$\dim W + \dim W^\perp = \dim V.$$

Proof. We want to use rank nullity so let's define a map. Define $\phi : V \rightarrow W^*$ by $\phi(u) = m_u \upharpoonright W$. From the previous cor this map is surjective. Then

$$\ker \phi = \{u \in V : m_u(w) = 0, \text{ for all } w \in W\} = W^\perp.$$

Then from rank nullity

$$\begin{aligned} \dim V &= \dim \text{im } \phi + \dim \ker \phi \\ &= \dim W^* + \dim W^\perp \end{aligned}$$

and we may identify $\dim W^* = \dim W$ since the space is finite dimensional. $\square$

3 A related problem (Algebra qual fall 2018 Q8)

Statement. Let V be a vector space over a field F and V^* its dual. A symmetric bilinear form $(\cdot, \cdot)$ on V is a map $V \times V \rightarrow F$ satisfying the rules

$$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w) \quad \text{and} \quad (v, w) = (w, v)$$

for $a, b \in F$ and $v_1, v_2, v, w \in V$. The form is nondegenerate if the only element w satisfying $(v, w) = 0$ for all $v \in V$ is $w = 0$.

Suppose $(\cdot, \cdot)$ is a symmetric nondegenerate bilinear form on V . If W is a subspace of V , define

$$W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}.$$

(a) Show that if X and Y are subspaces of V with $Y \subset X$, then $X^\perp \subseteq Y^\perp$.

(b) Define an injective linear map $\psi : Y^\perp/X^\perp \hookrightarrow (X/Y)^*$ which is an isomorphism if V is finite-dimensional.

Solution. (a) Suppose $v \in X^\perp$ that is $(v, x) = 0$ for all $x \in X$. Then if $y \in Y \subseteq X$ we must also have $(v, y) = 0$ so $v \in Y^\perp$.

(b) For each $v \in Y^\perp$ we associate the map $m_v : X/Y \rightarrow F$ defined by

$$m_v(u + Y) = (v, u) \in F$$

then $\phi(v) = m_v$. First why is this well defined. Suppose $u + Y = u' + Y$ so that $u = u' + y$. Then since $v \in Y^\perp$ we have

$$\begin{aligned} m_v(u + Y) &= (v, u) = (v, u' + y) = (v, u') + (v, y') \\ &= (v, u') + 0 = m_v(u' + Y). \end{aligned}$$

Then linearity follows from the bilinearity of $(\cdot, \cdot)$.

Now what is the kernel of this map. If $v \in \ker \phi$ then

$$m_v(u + Y) = (v, u) = 0, \text{ for all } u \in X$$

hence $v \in X^\perp$. Then the first isomorphism theorem yields the induced map $\psi : Y^\perp/X^\perp \rightarrow (X/Y)^*$ is injective.

To show this map is an isomorphism note that when the space is finite dimension so is the quotient (X/Y) thus it is isomorphic to its dual and

$$\dim (X/Y)^* = \dim (X/Y) = \dim X - \dim Y.$$

Regarding $\text{im } \psi$ as a subspace of $(X/Y)^*$ to show the map is surjective it suffices to show the image (and also source since the map is injective) has the same dimension $\dim X - \dim Y$. Then we have

$$\dim (Y^\perp/X^\perp) = \dim Y^\perp - \dim X^\perp.$$

Then from the above we have

$$\begin{aligned} \dim Y^\perp - \dim X^\perp &= (\dim V - \dim Y) - (\dim V - \dim X) \\ &= \dim X - \dim Y \end{aligned}$$

so the dimensions match as desired. □