Non-degenerate forms and dual spaces

1 Dual space representation theorem

Let V be a vector space over a field k.

Definition. A symmetric bilinear form on V is a bilinear map $(\cdot,\cdot) : V \times V \to k$ satisfying

$$(u, v) = (v, u), \text{ for all } u, v \in V.$$

The form is called **non-degenerate** if

$$(u, v) = 0, \text{ for all } v \in V \text{ implies } u = 0.$$

From now on equip V with a symmetric form $(\cdot,\cdot)$.

Definition. For each $u \in V$ we define the map $m_u : V \to k$, by

$$m_u(v) = (u, v).$$

Proposition. The map sending $u \mapsto m_u$ is linear map from $V \to V^*$.

Proof. By definition $m_u \in V^*$. For linearity it suffices to check that

$$m_{au} = a m_u$$
$$m_{u+v} = m_u + m_v$$

both of which follow from linearity in the first coordinate for $(\cdot,\cdot)$. □

Theorem (Representation theorem for V^*)**.** Let V be finite dimensional, and suppose the bilinear form is non-degenerate. Then for every $f \in V^*$ there is a $u \in V$ such that

$$f(v) = (u, v), \text{ for all } v \in V.$$

That is all functionals are induced from the bilinear form. More formally the map $\psi(u) = m_u$ is an isomorphism.

Proof. Recall that the map $\psi : V \to V^*$ defined by $\psi(u) = m_u$ is linear. Then the first isomorphism theorem yields

$$\frac{V}{\ker \psi} \cong \operatorname{im} \psi \subseteq V^*.$$

Now consider that $\psi(u) = m_u$ is the zero map iff

$$m_u(v) = (u, v) = 0, \text{ for all } v \in V.$$

Then since the form is non-degenerate this implies $u = 0$. Then ψ is injective and identifies V is a subspace of V^*. But then since V is finite dimensional $V \cong V^*$ so ψ must be surjective, therefore an isomorphism. □

1.1 Representation restricts to a subspace

Proposition. Let V be finite dimensional and let $W \subseteq V$ be a subspace. Then we may identify W^* with a subspace of V^*.

Proof. Let $\{w_i\}_{i \in I}$ be a basis for W. Since W is a vector space we may define a map by defining its image on a basis. Then consider the maps $f_i : W \to k$ defined by

$$f_i(w_j) = \begin{cases} 1, & i = j \\ 0, & \text{else} \end{cases}.$$

It's straight forward to check these form a basis for the dual space W^* (from linear combinations of these maps we can construct any possible image for the basis $\{w_i\}_{i \in I}$ that we want, and any functional is determined by its image on this basis.) Now extend $\{w_i\}$ to a basis for V by $\{w_i\}_{i \in I} \cup \{v_j\}_{j \in J}$. Then a basis for V^* is

$$\{f_i\}_{i \in I} \cup \{g_j\}_{j \in J}$$

thus we have $W^* = \operatorname{span}\{f_i\} \subseteq \operatorname{span}(\{f_i\} \cup \{g_j\}) = V^*$. □

Corollary. When V is finite dimensional, the form is non-degenerate, and $W \subseteq V$ is a subspace the map $\phi(u) = m_u \restriction W$ is surjective onto W^*.

Proof. If $\{f_i\}_{i \in I}$ is a basis for W^* then each f_i extends to a functional on V^* (since the basis for W extends to a basis for V.) Then there is $u_i \in V$ such that $f_i = m_{u_i}$ for each $i \in I$. Then restricting $f_i = m_{u_i} \restriction W$ so we obtain a basis for the dual space W^* and the map is surjective. □

2 Dimension of orthogonal subspace.

Definition. Let $W \subseteq V$ be a subspace. Then

$$W^{\perp} = \{u \in V : (u, w) = 0, \text{ for all } w \in W\}.$$

Proposition. If V is finite dimensional and the bilinear form is non-degenerate then

$$\dim W + \dim W^{\perp} = \dim V.$$

Proof. We want to use rank nullity so lets define a map. Define $\phi : V \to W^*$ by $\phi(u) = m_u \restriction W$. From the previous cor this map is surjective. Then

$$\ker \phi = \{u \in V : m_u(w) = 0, \text{ for all } w \in W\} = W^{\perp}.$$

Then from rank nullity

$$\begin{aligned} \dim V &= \dim \operatorname{im} \phi + \dim \ker \phi \\ &= \dim W^* + \dim W^{\perp} \end{aligned}$$

and we may identify $\dim W^* = \dim W$ since the space is finite dimensional. □

3 A related problem (Algebra qual fall 2015 Q8)

Let V be a vector space over a field F and V^* its dual. A symmetric bilinear form $(\cdot, \cdot)$ on V is a map $V \times V \to F$ satisfying the rules

$$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w) \quad \text{and} \quad (v, w) = (w, v)$$

for $a, b \in F$ and $v_1, v_2, v, w \in V$. The form is nondegenerate if the only element w satisfying $(v, w) = 0$ for all $v \in V$ is $w = 0$.
Suppose $(\cdot, \cdot)$ is a symmetric nondegenerate bilinear form on V. If W is a subspace of V, define

$$W^{\perp} = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}.$$

(a) Show that if X and Y are subspaces of V with $Y \subset X$, then $X^{\perp} \subseteq Y^{\perp}$.
(b) Define an injective linear map $\psi : Y^{\perp}/X^{\perp} \hookrightarrow (X/Y)^*$ which is an isomorphism if V is finite-dimensional.

Proof. **(a)** Suppose $v \in X^{\perp}$ that is $(v, x) = 0$ for all $x \in X$. Then if $y \in Y \subseteq X$ we must also have $(v, y) = 0$ so $v \in Y^{\perp}$.

(b) For each $v \in Y^{\perp}$ we associate the map $m_v : X/Y \to F$ defined by

$$m_v\left(u+Y\right) = (v,u) \in F$$

then $\phi\left(v\right) = m_v$. First why is this well defined. Suppose $u + Y = u' + Y$ so that $u = u' + y$. Then since $v \in Y^{\perp}$ we have

$$\begin{aligned} m_v\left(u+Y\right) = (v,u) = \left(v,u'+y\right) &= \left(v,u'\right) + \left(v,y'\right) \\ &= \left(v,u'\right) + 0 = m_v\left(u'+Y\right). \end{aligned}$$

Then linearity follows from the bilinearity of $(\cdot,\cdot)$.

Now what is the kernel of this map. If $v \in \ker\phi$ then

$$m_v\left(u+Y\right) = (v,u) = 0, \text{ for all } u \in X$$

hence $v \in X^{\perp}$. Then the first isomorphism thoerem yields the induced map $\psi : Y^{\perp}/X^{\perp} \to (X/Y)^*$ is injective.

To show this map is an isomorphism note that when the space is finite dimension so is the quotient (X/Y) thus it is isomorphic to its dual and

$$\dim\left(X/Y\right)^* = \dim\left(X/Y\right) = \dim X - \dim Y.$$

Regarding im ψ as a subspace of $(X/Y)^*$ to show the map is surjective it suffices to show the image (and also source since the map is injective) has the same dimension $\dim X - \dim Y$. Then we have

$$\dim\left(Y^{\perp}/X^{\perp}\right) = \dim Y^{\perp} - \dim X^{\perp}.$$

Then from the above we have

$$\begin{aligned} \dim Y^{\perp} - \dim X^{\perp} &= (\dim V - \dim Y) - (\dim V - \dim X) \\ &= \dim X - \dim Y \end{aligned}$$

so the dimensions match as desired. □