Upper Bound for Longest Path in Polynomial Divisor Graphs

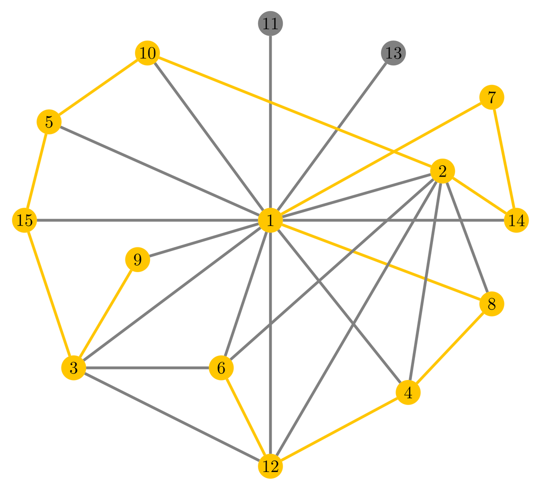
Jonathan Parlett[1] and Nicole Froitzheim[2]
with Jay Calkins[3], Kayla Traxler[4], Dr. Angel Kumchev[3] and Dr. Nathan McNew[3]

[1]Drexel University, [2]CUNY Baruch, [3]Towson University, [4]UW - La Crosse

Introduction

What is the longest sequence of distinct integers, at most n, such that for any adjacent numbers in the sequence one divides the other? The length of this sequence, denoted $f(n)$, has been well studied by Erdős, Freud, Hegyvari, Tenenbaum, and Saias and may be equivalently stated as finding the length of a longest path in the divisor graph $D(n)$.

Integer Divisor Graph



The graph $D(n)$ has vertices $\{1, 2, \dots, n\}$ with an edge between $s, t \le n$ if $s \mid t$ or $t \mid s$. The graph $D(15)$ is pictured with longest path in gold.

Eric Saias first obtained the best qualitative bounds on $f(n)$, showing there exist constants $c_2 > c_1 > 0$ such that for n sufficiently large

$$c_1 \frac{n}{\log n} \le f(n) \le c_2 \frac{n}{\log n}.$$

We show an analogous result for monic polynomials over a finite field. Let $\mathbb{F}_q$ be the finite field of order q, and let $f_q(n)$ denote the length of the longest sequence of distinct monic polynomials in $\mathbb{F}_q[x]$ with degree at most n such that for any adjacent polynomials one divides the other. Then we have the following result.

Theorem

There exists a constant k_q such that for n sufficiently large

$$f_q(n) \le k_q \frac{q^n}{n}.$$

Function Fields

The function field $\mathbb{F}_q[x]$ is similar to $\mathbb{Z}$ in many ways, and has been the source of inspiration for many number theoretic questions. While there is not a strict correspondence, many analogies exist between the two.

Integers	Polynomials
$n \in \mathbb{N}$	monic $F \in \mathbb{F}_q[x]$
$\log n$	$\deg F$
$\lvert n \rvert$	$\lvert\lvert F \rvert\rvert = q^{\deg F}$
Primes, p	Irreducibles, P

Numerous results are easier to show for function fields, leading to greater understanding of the integer case. For example, the abc-conjecture, and the Riemann Hypothesis have been proven for function fields.

Polynomial Divisor and LCM Graphs

Let $\mathcal{M}_q(n) := \{F \in \mathbb{F}[x] : F \text{ monic}, \ \deg F \le n\}$, and let $[F, G]$ denote the least common multiple (LCM) of the polynomials F, G. Then we define the polynomial divisor and LCM graphs over $\mathcal{M}_q(n)$ as follows:

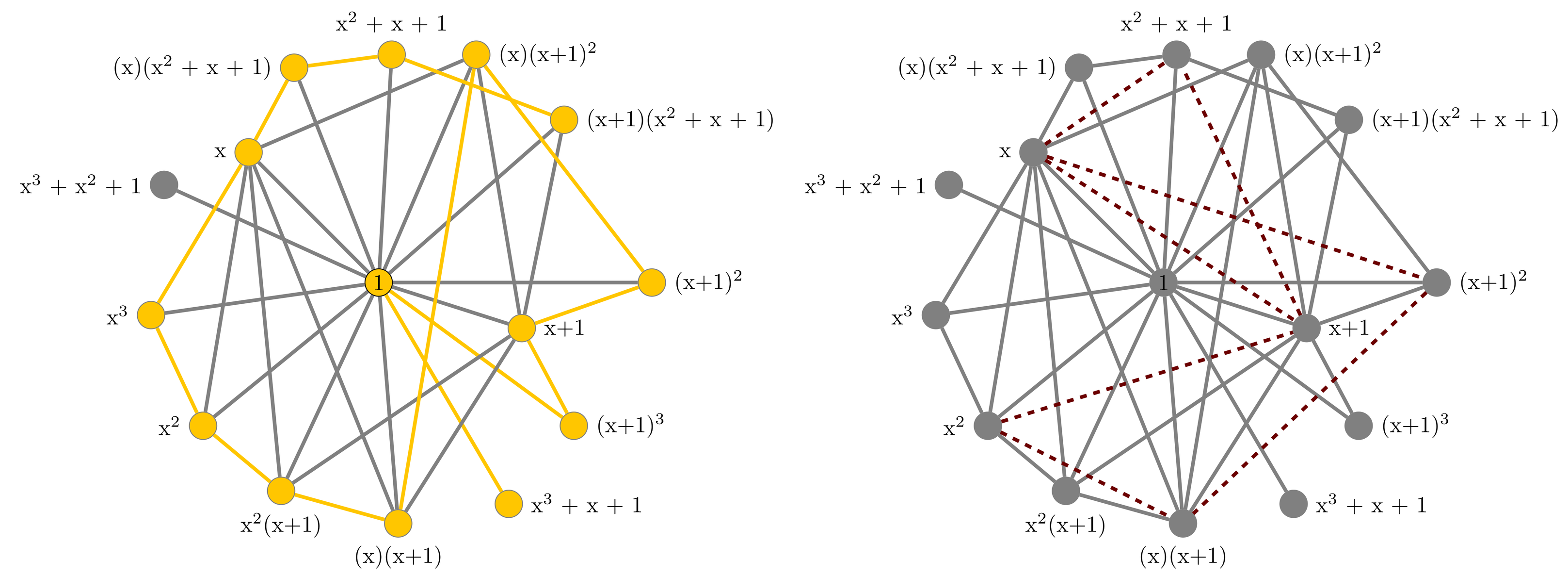
Definition

Polynomial Divisor Graph $D_q(n)$**:** The graph with vertices $\mathcal{M}_q(n)$ with an edge between F, G if $F \mid G$ or $G \mid F$.

Polynomial LCM Graph $L_q(n)$**:** The graph with vertices $\mathcal{M}_q(n)$, with an edge between F, G if $\deg[F, G] \le n$.

We may equivalently define $f_q(n)$ as the length of the longest path in $D_q(n)$. Note that $D_q(n)$ is a subgraph of $L_q(n)$ since if $G \mid F$ then $\deg[F, G] = \deg F \le n$. As a consequence, to bound $f_q(n)$, it suffices to bound the size of the longest path in $L_q(n)$.

Polynomial Graphs



Left: Divisor graph for $\mathcal{M}_2(3)$ with longest path colored gold. **Right:** LCM graph for $\mathcal{M}_2(3)$ with edges exclusive to LCM graph in dashed red.

Divisor Gaps and The Schinzel-Szkeres Function

The number of integers with a bounded divisor gap constrains $f(n)$ in the integer case [1, 2], and the same is true for the polynomial case. We exploit the work of Weingartner [3], counting the number of polynomials with a bounded divisor gap, to obtain our bound on $f_q(n)$.

Definition

(a) The divisor sequence of $F \in \mathcal{M}_q(n)$ is the increasing sequence $\{\deg D_i\}_{i=1}^{\tau(F)}$ where $D_i \mid F$. The divisor gap of F is the largest difference in adjacent terms, $\operatorname{gap} F := \max_{1 \le i < \tau(F)}(\deg D_{i+1} - \deg D_i)$.

(b) Let $\delta^-(F)$ be the smallest degree of an irreducible factor of $F \in \mathcal{M}_q(n)$. We define the (polynomial) Schinzel-Szekeres function by $\Phi(F) := \max\{\deg D + \delta^-(D) : D \mid F\}$.

Example: The polynomial x^4 has divisors $1, x, x^2, x^3, x^4$, and therefore divisor sequence $0, 1, 2, 3, 4$, and thus $\operatorname{gap} x^4 = 1$. We have $\Phi(x^4) = \deg x^4 + \delta^-(x^4) = 4 + 1 = 5$. We note that $\operatorname{gap} x^4 = \Phi(x^4) - \deg x^4$.

Weingartner [3] establishes that $\operatorname{gap} F = \Phi(F) - \deg F$ in general, and bounds the size of the set

$$\mathcal{A}(n) := \{F \in \mathcal{M}_q(n) : \Phi(F) \le n\}$$

above by $c\frac{q^n}{n}$ for some constant c.

Upper-Bound Sketch

Let $\mathcal{P}$ be a longest path in $L_q(n)$. We define the set

$$\mathcal{B}(n) := \{F \in \mathcal{M}_q(n) : F \notin \mathcal{A}(n) \text{ but any proper divisor } D \in \mathcal{A}(n)\}.$$

We will use the sets $\mathcal{A}(n), \mathcal{B}(n)$ to partition our path.

- ◇ From Weingartner [3]: $\quad |\mathcal{P} \cap \mathcal{A}(n)| \le |\mathcal{A}(n)| \le c\frac{q^n}{n}$.
- ◇ Next, we break $\mathcal{P} \setminus \mathcal{A}(n)$ into subpaths, $\mathcal{S}$, and show that for each $\mathcal{S}$ there is $B \in \mathcal{B}(n)$ that divides all $F \in \mathcal{S}$. Letting $\mathcal{S}(B)$ denote the set of subpaths with a fixed B value, we may write
$$|\mathcal{P} \setminus \mathcal{A}(n)| = \sum_{B \in \mathcal{B}(n)} \sum_{\mathcal{S} \in \mathcal{S}(B)} |\mathcal{S}|.$$
- ◇ A collection of subpaths with the same B is in bijection with a collection of paths in the smaller LCM graph $L_q(n - \deg B)$. Thus, we can bound $\sum_{\mathcal{S} \in \mathcal{S}(B)} |\mathcal{S}|$ in two ways: either by the number of multiples of B, or by the number of vertices of $L_q(n - \deg B)$ that we can cover with $|\mathcal{B}(n)|$ paths.
- ◇ It suffices to break our sum into parts when either B has large degree or there are many subpaths $\mathcal{S}$ with that fixed B, and the complementary case where both of these quantities are small. In either scenario, the sum above is bounded by $c\frac{q^n}{n}$ yielding our theorem.

Future Work

It is likely possible to obtain an explicit value for the constant k by conducting a tighter analysis of many of the quantities used in the proof. Future work would progress towards this goal.

Acknowledgements

This project is funded by the National Science Foundation and Towson University. Special thanks to our project mentors Dr. Kumchev and Dr. McNew as well as the Towson University REU PI Dr. Cornwell.

References

[1] Eric Saias. Applications des entiers à diviseurs denses. *Acta Arithmetica*, 83(3):225–240, 1998.

[2] Gérald Tenenbaum. Sur un probl'eme de crible et ses applications, 2. corrigendum et t'etude du graphe divisoriel. *Annales scientifiques de l'École Normale Supérieure*, 28(2):115–127, 1995.

[3] Andreas Weingartner. On the degrees of polynomial divisors over finite fields. *Mathematical Proceedings of the Cambridge Philosophical Society*, 161(3):469–487, 2016.

LaTeX TikZposter