Long Paths in Polynomial Divisor Graphs

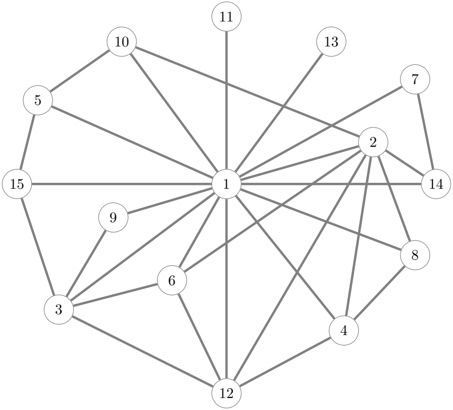
YMC 2024

Jonathan Parlett Kayla Traxler

(With mentors Dr. Angel Kumchev, Dr. Nathan McNew, and collaborators Jay Calkins and Nicole Froitzheim.)

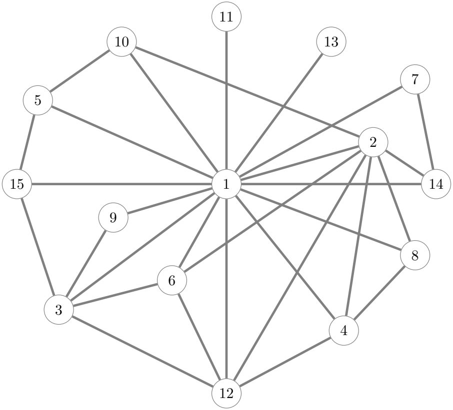
Divisor Graphs

A divisor graph, $D(n)$ contains vertices $\{1, 2, ..., n\}$ and an edge between two vertices, u and v, if $u|v$ or $v|u$.



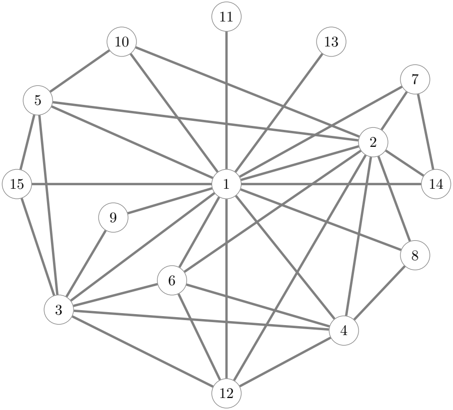
Divisor Graphs

For this talk $f(n)$ is the length of the longest path in $D(n)$.



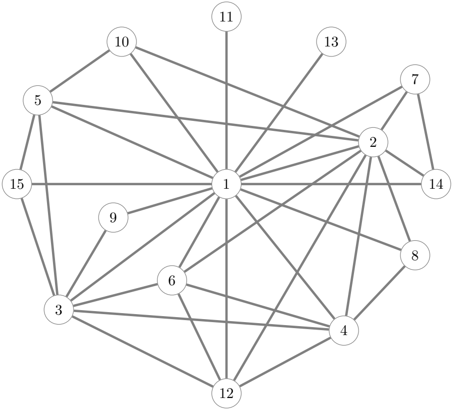
LCM Graphs

An LCM graph, $L(n)$ contains vertices $\{1,2,\ldots,n\}$ and an edge between two vertices, u and v, if $[u,v] \leq n$.



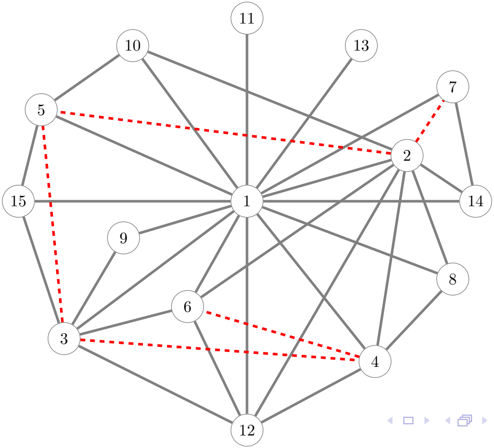
LCM Graphs

Similarly let $g(n)$ denote the length of the longest path in $L(n)$.



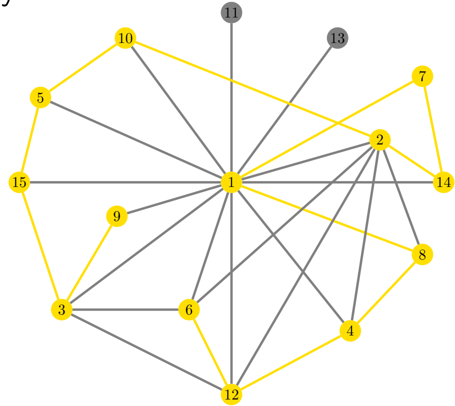
LCM Graphs

Note: $D(n)$ is a subgraph of $L(n)$ because if $u|v$, $[u,v] = v \leq n$. Thus $f(n) \leq g(n)$.



A long path in $D(n)$

The example below shows that $f(15) \geq 13$, and further analysis shows this is the best we can do.



Hint: If we must start at 11 or 13 how long can our path be?

Long Paths

$f(n)$ has been studied previously by Pollington, Pomerance, Tenenbaum, and Saias using analytic techniques.

In particular Eric Saias obtained the best known bounds

Theorem

For sufficiently large n there exist constants c_1, c_2 s.t

$$c_1 \frac{n}{\log n} \leq f(n) \leq g(n) \leq c_2 \frac{n}{\log n}.$$

Long Paths

Building upon the work of these previous authors we obtain an analgous result for polynomials over a finite field.

In order to explain what that means we'll need some notation.

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Let $\mathbb{F}_q$ be a finite field of order q. For example $\mathbb{Z}_2 = \{0, 1\}$ with addition and multiplication mod 2 is the finite field of order 2.

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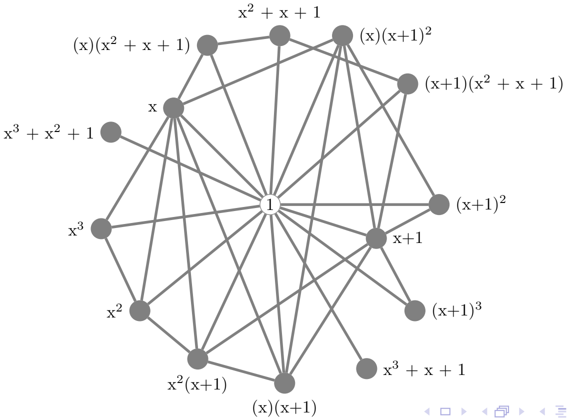
We denote the polynomials with degree at most n by

$$\mathcal{M}_q^{\leq n} = \{\text{monic } F \in \mathbb{F}_q[x] : \deg F \leq n\}$$

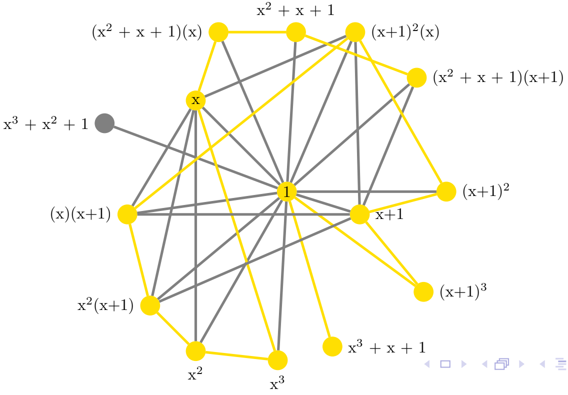
Polynomial Divisor Graphs

The Polynomial Divisor Graph $D_q(n)$ has vertices $\mathcal{M}_q^{\leq n}$. Below is the case $D_2(3)$ with vertices $\mathcal{M}_2^{\leq 3}$.



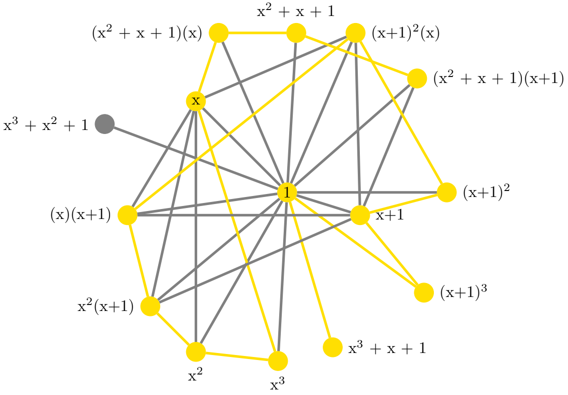
Polynomial Divisor Graphs

Similar to the integer case let $f_q(n)$ be the length of the longest path in $D_q(n)$. The below example shows that $f_2(3) \geq 14$.



Polynomial Divisor Graphs

Since we cannot include both the irreduciibles x^3+x+1 and x^3+x^2+1 we conclude $f_2(3)=14$.



cannot include in any path longer then 3

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Draw an edge between vertices $F, G \in \mathcal{M}_q^{\leq n}$ if $\deg[F, G] \leq n$.

With $g_q(n)$ the length of the longest path in $L_q(n)$ We still have $f_q(n) \leq g_q(n)$.

Result

Our main result is analogous to that of Saias.

Theorem 2024+

For sufficiently large n there exist constants c_1, c_2 s.t

$$c_1 \frac{q^n}{n} \leq f_q(n) \leq g_q(n) \leq c_2 \frac{q^n}{n}.$$

We will give ideas of the proof of this result shortly, but first some polynomial propaganda.

Polynomials (Why do we care?)

Polynomials over a finite field have similar properties to the integers.

Integers	Polynomials
$n \in \mathbb{N}$	monic $F \in \mathbb{F}_q[x]$
$\log n$	$\deg F$
$\lvert n \rvert = n$	$\lvert\lvert f \rvert\rvert = q^{\deg F}$
Primes, p	Irreducible polynomials, P
$\pi(n) = \# \{p \leq n : p \text{ prime}\}$	$\pi_q(n) = \# \left\{ \substack{P \in \mathcal{M}_q^n: \\ P \text{ irreducible}} \right\}$
Unique factorization into primes	Unique factorization into irreducibles

The Schinzel–Szekeres function

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$$\Phi(F) = \max\{\deg G + \delta^-(G) : G|F\}.$$

$\Phi(F)$ has connections to many important questions in number theory.

Divisor Gaps

Question: When does $F \in \mathcal{M}_q^n$ have a divisor of every degree up to n?

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For example, over $\mathbb{Z}_2$, x^4 does while x^2+x+1 does not.

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Answer: Exactly when $\Phi(F) - \deg F \leq 1$.

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This difference is also called the divisor gap of F.

Divisor Gaps

Andreas Weingartner provides estimates for

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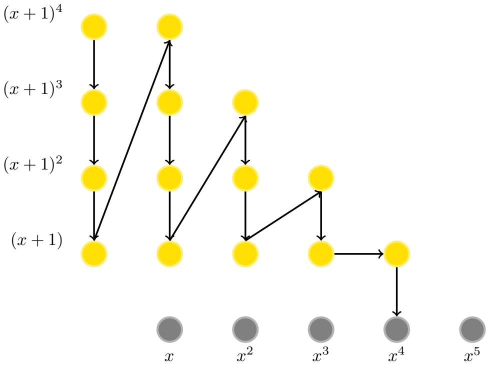
Now we will show the main ideas of the proofs starting with the lower bound.

Dot Diagram

The following base case represents $\Gamma(6, x+1)$. The path is constructed by connecting multiples of irreducible polynomials that are adjacent in our ordering.



Polynomial Path

Our polynomial path is

$$\Gamma(d,P) = \begin{cases} 1 \to \Gamma_0 \to \Gamma_1 \to \dots \\ \to \Gamma_b \to^* \Gamma(d,P^\dagger) & P \le P_d \text{ and } b = a^\dagger \\ 1 \to \Gamma_0 \to \Gamma_1 \to \dots \to \Gamma_b \\ \to PP^{\dagger a^\dagger} \to^* \Gamma(d,P^\dagger) & P \le P_d \text{ and } b \ne a^\dagger \\ \Gamma(d,P_d) & P > P_d \end{cases}$$

Intro/Induction

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Base cases are $\Gamma(d,x)$ and $\Gamma(d,x+1)$.

Assume the conditions hold for any irreducibles $< P$.

Lower Bound Proof: Conditions

$$\Gamma(d,P) = 1 \to P^{a_0} \to \cdots \to x$$

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3. $\Gamma(d,P)$ consists of all polynomials F such that $P^+(F) \leq P$ and $\Phi(F) \leq d-1$.

Lower Bound Proof: Counting $\Gamma(d,P)$

Since our path $\Gamma(n,P)$ contains all polynomials $F \in \mathcal{M}_q^{\leq n}$ such that $\Phi(F) \leq n-1$ we have that

$$\Gamma(n,P) \supseteq \{F \in \mathcal{M}_q^{n-2} : \Phi(F) \leq n-1\}.$$

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Using the results of Weingartner discussed previously we can show that

$$|\Gamma(n, P)| \geq \#\{F \in \mathcal{M}_q^{n-2} | \Phi(F) \leq n-1\}$$
$$\geq q^{n-2} \cdot \frac{c}{n-1} \geq c' \frac{q^n}{n}.$$

Where $c' = \frac{c}{q^2}$

Upper-Bound: $\mathcal{A}(n), \mathcal{B}(n)$

$\mathcal{A}(n) = \{F \in \mathcal{M}_q^{\leq n} : \Phi(F) \leq n\}$. This set is closely related to the set of polynomials with divisor gap at most m.

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$\mathcal{B}(n) = \left\{ \substack{F \in \mathcal{M}_q^{\leq n} : F \notin \mathcal{A}(n) \\ \text{but any proper divisor } G \text{ of } F \text{ in } \mathcal{A}(n)} \right\}$.

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If $B_1 \neq B_2 \in \mathcal{B}(n)$, then $\deg[B_1, B_2] > n$.

Upper-bound: Sketch

Let $\mathcal{P} = (F_1, F_2, \ldots, F_{g_q(n)})$ be a longest path in $L_q(n)$.

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$$|\mathcal{P} \cap \mathcal{A}(n)| \leq |\mathcal{A}(n)| \leq c\frac{q^n}{n}$$

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Let $\mathcal{S}(\mathcal{P})$ be the set of all subpaths.

$$\left|\mathcal{P}\setminus\mathcal{A}(n)\right| = \sum_{\mathcal{S}\in\mathcal{S}(\mathcal{P})} |\mathcal{S}|$$

Upper-bound: Subpaths $\mathcal{S}$

For each $\mathcal{S} = (F_i, F_{i+1}, \ldots, F_j)$, since $F_k \notin \mathcal{A}(n)$, there is $B_k | F_k$.

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Then since the LCMs of things in $\mathcal{B}(n)$ are large $(> n)$, we know all these B_k are equal

$$\deg[B_k, B_{k+1}] \leq \deg[F_k, F_{k+1}] \leq n$$

Upper-bound: Subpaths $\mathcal{S}$

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So for each $\mathcal{S}$, there is a $B(\mathcal{S}) \in \mathcal{B}(n)$ that divides everything in the subpath.

Upper-bound: Sketch

$$\left|\mathcal{P} \setminus \mathcal{A}(n)\right| = \sum_{\mathcal{S} \in \mathcal{S}(\mathcal{P})} |S| = \sum_{B \in \mathcal{B}(n)} \sum_{\substack{\mathcal{S} \in \mathcal{S}(\mathcal{P}) \\ B(\mathcal{S}) = B}} |\mathcal{S}|$$

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We can bound the inner sum $\sum_{\substack{\mathcal{S} \in \mathcal{S}(\mathcal{P}) \\ B(\mathcal{S}) = B}} |\mathcal{S}|$ in two different ways.

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Since every vertex in these subpaths is divisible by B,

$$\sum_{\substack{\mathcal{S}\in\mathcal{S}(\mathcal{P})\\ B(\mathcal{S})=B}} |\mathcal{S}| \leq \#(\text{multiples of } B) \leq q^{n-\deg B}$$

Upper-bound: Sketch

The 2nd way requires an observation. Each of these $\mathcal{S}$ implies a path in the LCM graph $L_q(n - \deg B)$

$$\mathcal{S} = (F_i, F_{i+1}, \ldots, F_j) = (BG_i, BG_{i+1}, \ldots, BG_j)$$

So $(G_i, G_{i+1}, \ldots, G_j)$ is a path in $L_q(n - \deg B)$.

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So $(G_i, G_{i+1}, \dots, G_j)$ is a path in $L_q(n - \deg B)$. Let $K(B) = \left|\{\mathcal{S} \in \mathcal{S}(\mathcal{P}) : B(\mathcal{S}) = B\}\right|$. The number of subpaths with a fixed B value.

Upper-bound: Sketch

$$\sum_{\substack{\mathcal{S}\in\mathcal{S}(\mathcal{P})\\ B(\mathcal{S})=B}} |\mathcal{S}| \leq \#\begin{pmatrix}\text{vertices we can cover in } L_q(n-\deg B)\\ \text{with } K(B) \text{ paths}\end{pmatrix} \leq \frac{q^{n-\deg B}}{n-\deg B}\log K(B)$$

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We can break up our sum strategically so that each of our bounds is as small as possible.

Upper-bound: Sketch

$$\sum_{B \in \mathcal{B}(n)} \sum_{\substack{\mathcal{S} \in \mathcal{S}(\mathcal{P}) \\ B(\mathcal{S}) = B \\ K(B), \deg B \text{ are small}}} |\mathcal{S}| + \sum_{B \in \mathcal{B}(n)} \sum_{\substack{\mathcal{S} \in \mathcal{S}(\mathcal{P}) \\ B(\mathcal{S}) = B \\ \text{one of } K(B), \deg B \text{ is big}}} |\mathcal{S}|$$

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When we don't have too many subpaths, we can use our path bound. In the other case, we use our multiples bound.

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Thus we obtain sums of the form

$$\sum_{\substack{A \in \mathcal{A}(n) \\ \text{+other constraints}}} h(n, A)$$

where h depends on n and A only.

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We are currently finishing up bounding these sums using the results of Weingartner.

Acknowledgements

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- Our collaborators Nicole Froitzheim, and Jay Calkins.
- The NSF for funding this research.
- The YMC organizers for inviting us to give this talk!

Thank you!